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THE EQUILIBRIUM OF A STAR

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

THE EQUILIBRIUM OF A STAR

H.Y. Chiu

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II.1. Introduction

The light from most stars does not seem to vary much with time, this is an indication that a quasi-static mechanical equilibrium condition persists in stars. Our sun, which has been observed with great precision, shows mechanical motion only in the outermost layer whose mass is a negligible fraction of the total mass. Therefore as a first approximation we may assume that in all stars a mechanical equilibrium is established in the bulk part of the star. The basis of this approximation is further assured by results of a complete dynamical treatment in Chapter , where it is shown explicitly that dynamical effects become important only when the velocity of moving matter of the star approaches the speed of sound, which is roughly the thermal velocity of gas particles, and also the velocity of free falling matter corresponding to the surface gravity of the stars.

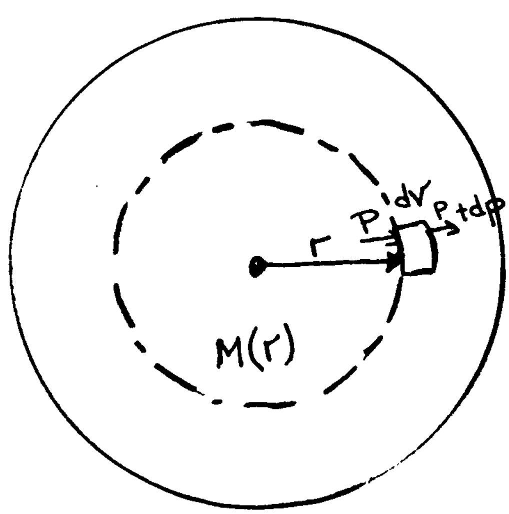
By a similar argument rotation can be neglected, as far as the mechanical structure is concerned, provided that the centrifugal force due to rotation is small compared with the gravitational acceleration at points under consideration.

Neglecting the motion of fluid elements and effects of rotation the structure of a star assumes spherical symmetry. The conditions for equilibrium become simple differential equations from which, approximate relations between the central temperature, central density, and the mass of star can be obtained. These relations are helpful to understanding the structure of the stars.

In addition to obtaining the condition for mechanical equilibrium for a star, we also develope a qualitative theory of stellar structure, based on methods of dimensional analysis. We shall show that, because of quantum effects, there exists a maximum temperature for stars of a given mass, and that there exists a maximum mass for stars at zero temperature.

II.2. Hydrostatic Equilibrium.

A star is held together by its own gravitational field. Since the gravitational field of a mass element is isotropic, neglecting anisotropic effects such as rotation and magnetic fields, the structure of a star assumes a spherical symmetry. Consider a volume element dv at a distance r from the center of the star. The gravitational force acting on the mass of the volume element is $-\rho \frac{GM(r)}{r^2} dv$, where M(r) is the mass enclosed within a sphere of symmetry of radius r and ρ is the density. (Fig. II.1).



The difference between the hydrostatic forces is dPdA where A is the area of the volume element and dP is the difference between the dydrostatic pressure at the two surfaces of the volume element. In equilibrium the gravitational force is balanced by the hydrostatic force. Hence

$$dP\,dA = \frac{dP}{dr}\,dv = -\rho\,\frac{GM(r)}{r^2}\,dv \qquad \text{(II.1)}$$

or:

$$\frac{dP}{dr} = -\rho\,\frac{GM(r)}{r^2} \qquad \text{(II.2)}$$

This is often referred to as the equation of hydrostatic equilibrium. By definition, $M(r) = \int_0^r 4\pi r^2 \rho\, dr$ and the corresponding differential equation is

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho \qquad \text{(II.3)}$$

the associated boundary condition is $M(r) = 0$ at r=0.

Eqs. (II.2) and (II.3) are the two basic equations of stellar structure. Generally P is a function of two thermodynamic variables which may be chosen to be ρ and T, where T is the temperature. Therefore solutions of Eqs. (II.2) and (II.3) are not unique. Additional equations are required to render uniqueness. The additional equations are energy transfer and energy production equations. They are discussed in Chapter V. The nature of the complete set of equations are discussed in Chapter IX.

II.3. The Virial Theorem

Many important properties of stellar structure deal with the mechanical equilibrium. Most of these properties can be obtained from an integral theorem known as the virial theorem. The virial theorem is an integral representation of the hydrostatic equilibrium condition, it relates the self-gravitational energy of a star to its thermodynamic energy.

Multiply both sides of Eq. (II.2) by $\frac{4\pi r^3}{3} dr$, we find

$$\frac{4\pi}{3} r^3 dp = -\frac{1}{3} \cdot 4\pi r^3 \rho \frac{GM(r)}{r} dr \tag{II.4}$$

Denote by V the quantity $\frac{4\pi}{3} r^3$. V is the volume containing mass Mr. Substitute Eq. (II.3) into the right hand side of Eq. (II.4) we have

$$VdP = -\frac{1}{3} \frac{GM(r)}{r} dM(r) \tag{II.5}$$

Eq. (II.5) is integrated over the star. By partially integrating $\int VdP$, we obtain

$$\int V\,dP = PV \Big|_{\text{boundary}}^{\text{center}} - \int P dV \tag{II.6}$$

Since V vanishes at the center, P vanishes at the boundary and therefore the first term vanishes. Eq. (II.5) becomes

$$-3\int P\,dV = -\int \frac{GM_{(r)}}{r}\,dM_{(r)} \qquad \text{(II.7)}$$

This is the Virial theorem. The right hand side is the self-gravitational energy of the star; that is, the energy that is needed to disperse the material of the star into infinity. The left hand side is a kind of thermodynamic energy.

From our treatment on thermodynamics and statistical physics in chapter IV, it can be established that the pressure P is related to the thermodynamic energy density of the gas E by simple relations in the following two limiting cases of gas temperature*

$$P = \frac{2}{3}E \qquad \text{non relativistic gas (II.8)}$$

$$P = \frac{1}{3}E \qquad \text{relativistic gas (II.9)}$$

From Eq. (II.7) we find the relation between the total thermodynamic energy of the star E_t and the self-gravitational

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* A gas is relativistic (or non-relativistic) if the thermal velocity is $\sim c$ (or $<< c$) where c is the velocity of light. If the mass of the gas particles is m, then for a relativistic gas $kT >> mc^2$ and for a nonrelativistic gas $kT << mc^2$.

energy of a star E_G :

$$\int E\,dV = E_t = -\frac{1}{2}E_G = -\frac{1}{2}\int \frac{GM(r)}{r}\,dM(r) \qquad \text{(II.10)}$$

non relativistic case

$$E_t = -E_G \qquad \text{((II.11)}$$

relativistic case

The total energy of a star W may be defined to be the sum of E_t and E_G. We therefore find:

$$W = E_T + E_G = +\frac{1}{2}E_G = -E_t \qquad \text{(II.12)}$$

non relativistic case

$$W = 0 \qquad \text{(II.13)}$$

relativistic case *

We therefore obtain an important result that a star composed of relativistic gas is not gravitationally bound. As a star approaches relativistic conditions instability may arise. These instabilities are discussed in more detail in Chapter XIII.

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* Strictly speaking, if a star is composed of only <u>one</u> type of gas particles, when the gas becomes relativisic, general relativistic corrections to the stellar structure equations (II.2) and (II.3) also becomes important and the present theory is no longer applicable. However, in a star both electrons and nuclei contribute to pressure and it may happen that, because of the comparative smallness of the electron mass, the electron gas can become relativistic while general relativistic corrections to (II.2) and (II.3) are still unimportant.

The photon gas is a relativistic gas, hence instabilities can also occur in a star if the radiation pressure dominates over gas pressure. We shall show in Sect. II that this will give rise to a maximum mass limit for stars.

II.4. Secular Stability of the Stars.

The fact that a star made of relativistic gas is unstable leads to the important question of stability for the stars. There are many types of stability problems and criteria for stability, but here we only need to ask if the equilibrium configuration of a star is stable against small perturbations. This kind of stability is called secular stability.

Small perturbatiolns applied to stars can be regarded as adiabatic perturbations. This assumption is fully justified in most cases except near the outer layers of a star. The time scale for redistribution of energy in stellar interior is of the order of 10^3 years or more ($\sim 10^5$ years for the sun). The time scale for perturbations to propagate across the star is of the order of the period of pulsation, which ranges from a few hours to a few days.

The following stability statements can now be made: A star is secularly stable if it is stable against arbitrary adiabatic perturbations; a star is secularly unstable if it is unstable against any one mode of perturbation. In terms of energy, a star is stable if its equilibrium configuration is located at the absolute minimum on its energy surface.

This means that the first variation of W vanishes and the second variation of W is greater than zero. To summarize, we have

$W < 0$ (condition for gravitational binding) (II.14)

$(\delta W)_{ad} = 0$ (condition for hydrostatic equilibrium) (II.15)

$(\delta^2 W)_{ad} > 0$ (condition for secular stability) (II.16)

where the subscript "ad" refers to adiabatic conditions and δ denotes variation. The adiabatic conditon is also the same as the condition S=constant where S is the entropy. In the rest of this section all variations referred to are taken to be under the condition s=constant.

Let $v = \frac{1}{\rho}$ be the specific volume (volume per unit mass) and let ϵ be the thermodynamic energy per unit mass. The variation on E_t gives

$$\delta E_t = \delta \int E\,dV = \delta \int_0^M \epsilon\, dM_{(r)} = \int_0^M \delta\epsilon\, dM(r) \qquad \text{(II.17)}$$

According to Chapter III,

$$\delta\epsilon = \left(\frac{\partial \epsilon}{\partial v}\right)_s \delta v = -P\,\delta v \qquad \text{(II.18)}$$

Further, from Eq. (II.3)

$$v = \frac{1}{\rho} = \frac{dV}{dM_{(r)}} \qquad \text{(II.19)}$$

hence

$$\delta v = \frac{d\,\delta V}{dM(r)} = \frac{d}{dM_{(r)}}\left(4\pi r^2 \delta r\right) \qquad \text{(II.20)}$$

Eq. (II.17) then becomes

$$\delta E_t = \int_0^R \frac{dP}{dM_{(r)}} \cdot 4\pi r^2 \delta r \, dM_{(r)} \qquad \text{(II.21)}$$

The variation on E_G is

$$\delta E_G = -\int_0^R \delta \frac{GM_{(r)}}{r} dM_{(r)} = \int_0^R \frac{GM_{(r)}}{r^2} \delta r \, dM_{(r)} \qquad \text{(II.22)}$$

The condition for equilibrium (II.15) thus becomes:

$$\delta W = \delta\left(E_t + E_G\right) = \int_0^R \left(\frac{dP}{dM_{(r)}} + \frac{GM_{(r)}}{4\pi r^4}\right) \delta r \, dM_{(r)} = 0 \qquad \text{(II.23)}$$

or:

$$\frac{dP}{dM_{(r)}} = -\frac{GM_{(r)}}{4\pi r^4} \qquad \text{(II.24)}$$

Substituting Eq. (II.3) into Eq. (II.24), we obtain

$$\frac{dP}{dr} = -\frac{\rho G M(r)}{r^2}$$

which is the same as Eq. (II.2).

The second variation of ϵ is

$$\delta^2 \epsilon = \delta\left[\left(\frac{\partial \epsilon}{\partial v}\right)_s \delta v\right] = -\delta\left[P \delta v\right] \qquad \text{(II.25)}$$

$$= -\left(\frac{\partial P}{\partial v}\right)_s (\delta v)^2 - P \delta^2 v$$

We find, however,

$$\delta^2 v = \delta\left(\frac{d\,\delta V}{dM_{(r)}}\right) - \frac{d}{dM_{(r)}}\left[8\pi r\,(\delta r)^2\right] \tag{II.26}$$

and

$$(\delta r)^2 = \frac{(\delta V)^2}{12\pi r V} \tag{II.27}$$

Define the first adiabatic exponent Γ_1 by the equation:*

$$\Gamma_1 = -\frac{v}{P}\left(\frac{\partial P}{\partial v}\right)_s \tag{II.28}$$

Γ_1 has a simple meaning: In an adiabatic process the relation between P and v is

$$P v^{\Gamma_1} = \text{Constant} \tag{II.29}$$

The second variation of E_t is an integral of $\delta^2 \epsilon$ over dM_r. The second variation of E_G is just

$$\delta^2 E_G = -\int_0^R \frac{2GM(r)}{r^3}(\delta r)^2 \cdot dM_{(r)} \tag{II.30}$$

Substitute Eqs. (II.26) through (II.28)) into Eq. (II.25) and Eq. (II.30), integrate Eq. (II.25) over $dM(r)$ we find

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*The rest of adiabatic exponents are defined in Chapter III.

$$\delta^2 W = \int_0^R \left\{ \frac{P\Gamma_1}{v} \left(\frac{d\,\delta V}{dM(r)} \right)^2 - P \frac{d}{dM(r)} \left[8\pi r (\delta r)^2 \right] - \frac{2GM(r)}{r^2} (\delta r)^2 \right\} dM(r)$$

$$= \int_0^R \left\{ \frac{P\Gamma_1}{v} \left(\frac{d\,\delta V}{dM(r)} \right)^2 - P \frac{d}{dM(r)} \left[8\pi r (\delta r)^2 \right] + \frac{dP}{dM(r)} \cdot 8\pi r (\delta r)^2 \right\} dM(r) \quad \text{(II.31)}$$

By partially integrating the second term we find it is identical with the third term. Eq. (II.31) then becomes

$$\delta^2 W = \int_0^R \left\{ \frac{P\Gamma_1}{v} \left(\frac{d\,\delta V}{dM(r)} \right)^2 + \frac{4}{3} \frac{dP}{dM(r)} \frac{(\delta V)^2}{V} \right\} dM(r) \quad \text{(II.32)}$$

Change variable from Mr to V, we find, since $v = \frac{dV}{dM(r)}$,

$$\delta^2 W = \int_0^{V_R} \left[\left(\frac{d\,\delta V}{dV} \right)^2 P\Gamma_1 + \frac{4}{3} \frac{dP}{dV} \frac{(\delta V)^2}{V} \right] dV \quad \text{(II.33)}$$

where the upper limit of integration V_R is the volume of the star and is $\frac{4\pi R^3}{3}$, R is the radius of the star.

The stability criterion can now be stated in terms of δW: The star is stable if $\delta^2 W$ is greater than zero for <u>every</u> δV, the star is unstable if $\delta^2 W$ is less than zero for a δV.

Let $g = \delta V$. Following Dyson's treatment, we define a quantity γ:

$$\gamma = \frac{\int_0^{V_R} \Gamma_1 P \left(\frac{dg}{dV} \right)^2 dV}{-\int_0^{V_R} \frac{dP}{V\,dV} g^2\, dV} \quad \text{(II.34)}$$

Since $\frac{dP}{dV} < 0$, $P > 0$ and $\Gamma_1 > 1$, γ is always greater than zero. Comparing Eq. (II.34) and (II.33) we see that the condition for stability is that $\gamma > \frac{4}{3}$ for every g. This condition is easily reducible into an eigen value problem.

Define η by the following equation:

$$\eta = \int_0^{V_k} \left\{ \Gamma_1 P \left(\frac{dg}{dV}\right)^2 + \gamma \frac{dP}{dV} \frac{g^2}{V} \right\} dV \qquad \text{(II.35)}$$

If η is set to be zero, Eq. (II.35) is the same as Eq. (II.34). If the structure of the star is given then Γ_1 and P are functions of V. If g is also a given function of V then from Eq. (II.34) γ can be computed. Conversely, if γ is given a priori, then the equation for g can be obtained from Eq. (II.35) by requiring that the only form of g is admissible, if it gives $\eta = 0$. This is the same as to state that

$$\delta\eta = 0 \qquad \text{(II.36)}$$

From the variational principle, this results in the following equations:

$$\frac{d}{dV}\left(\Gamma_1 P \frac{dg}{dV}\right) = \frac{\gamma}{V}\left(\frac{dP}{dV}\right) g \qquad \text{(II.37)}$$

The boundary conditon for Eq. (II.37) is

$$g=0 \quad \text{at } V=0 \qquad \text{(condition at the center)} \qquad \text{(II.38)}$$

$$\Gamma_1 P \frac{dg}{dV} = 0 \quad \text{at } P=0 \qquad \text{(condition at the surface)} \qquad \text{(II.39)}$$

These boundary condition are physical requirements. Certainly δV must vanish if V is zero, and $\frac{dg}{dV}$ must not have singularities anywhere. Eq. (II.37) is a self-adjoint second order differential equation; it is known from the theory of differential equations that solutions to Eq. (II.37) which satisfies boundary conditions at two points in space exist only for a limited range and values of γ . This set of γ is called the eigen value of the differential equation. Let the minimum eigen value be γ_0 . Then the statement of stability states that, if $\gamma_0 > 4/3$ the star is stable and if $\gamma_0 < 4/3$ the star i s not stable.

Generally Γ_1 is a function of thermodynamic variables and is not a constant. No general solution has been found for Eq. (II.39), hence no general stability criterion can be given. Each problem must be treated separately. However, in many cases $\Gamma_1 \cong$ constant. In this case solution to Eq. (II.37) is easily obtained. The solution is:

$$\gamma_0 = \Gamma_1 \tag{II.40}$$

$$g = kV \tag{II.41}$$

where k is a constant. The stability condition $\delta^2 W > 0$ becomes simply

$$\delta^2 W = k^2 \int_0^{V_R} \left\{ P\Gamma_1 + 4/3 \frac{dP}{dV} \cdot V \right\} dV > 0 \tag{II.42}$$

Partially integrating the second term, Eq. (II.42) becomes

$$\delta^2 W = k^2 \int_0^{V_R} P(\Gamma_1 - 4/3)\, dV > 0 \tag{II.43}$$

A non relativistic gas has Γ_1 = 5/3 and is stable. A relativistic gas has Γ_1 = 4/3, and is not stable. For other reasons (e.g. internal degrees of freedom) Γ_1 may be below 4/3.

II.5. Integral Theorems.

The virial theorem is an integral of the equation for hydrostatic equilibrium; it relates the thermodynamic energy of a star to its gravitational energy. There are other similar integrals expressing inequalities between the central pressure, temperatue, to the mass and radius of a star. Some of these integral theorems are discussed below.

Theorem 1. (Chandrasekhar). If the density ρ does not increase outward, then

$$\frac{1}{2} G \left(\frac{4}{3}\pi\right)^{1/3} \bar{\rho}^{-4/3}(r)\, M^{2/3}(r) \leq P_c - P \leq \frac{1}{2} G \left(\frac{4}{3}\pi\right)^{1/3} \rho_c^{4/3} M^{2/3}(r) \qquad \text{(II.44)}$$

where

$$\bar{\rho}(r) = M(r) \Big/ \frac{4\pi}{3} r^3 \qquad \text{(II.45)}$$

is the mean density interior to r.

Proof: Integrating the hydrostatic equilibrium equation (II.2) from 0 to r, we find

$$P_c - P = \frac{G}{4\pi} \int_0^r \frac{M(r)\, dM(r)}{r^4} \qquad \text{(II.46)}$$

Eliminate r in the integral by using Eq. (II.45), we find

$$P_c - P = \frac{1}{4\pi}\left(\frac{4}{3}\pi\right)^{4/3} G \int_0^r \rho^{-4/3}(r)\, M^{-1/3}(r)\, dM(r) \tag{II.47}$$

If ρ does not increase outwards, $\bar{\rho}(r)$ does not increase outwards too. Replace $\bar{\rho}$ in the integral by ρ_c and $\bar{\rho}(r)$ respectively, we obtain the inequality (II.44).

Set r=R in Eq. (II.44), we find

$$\frac{1}{2} G \left(\frac{4}{3}\pi\right)^{1/3} \rho_c^{4/3} M^{2/3} \geq P_c \geq \frac{3}{8\pi}\frac{GM^2}{R^4} \tag{II.48}$$

Numerically the right hand side of Eq. (II.48) is:

$$P_c \geq 1.33\times 10^{13} \left(\frac{M}{\odot}\right)^2 \left(\frac{R_\odot}{R}\right)^4 \frac{d}{cm^2} \tag{II.49}$$

<u>Theorem 2.</u> (Ritter). In equilibrium,

$$\pi P_c R^3 + \frac{3}{8}\frac{GM^2}{R} > -E_G > \frac{GM^2}{2R} \tag{II.50}$$

<u>Proof</u>. Since

$$\frac{d}{dr}\left(P + \frac{GM^2(r)}{8\pi r^4}\right) = -\frac{GM^2(r)}{2\pi r^5} < 0 \tag{II.51}$$

we conclude that $P + \frac{GM^2(r)}{8\pi r^4}$ decreases outward.

$$P_c > P + \frac{GM^2(r)}{8\pi r^4} > \frac{GM^2}{8\pi R^4} \tag{II.52}$$

Multiply Eq. (II.52) by $3\pi r^2 dr$ and integrate from r=0 to r=R. We find

$$\pi P_c R^3 > \frac{3}{4}\int P\,dV - \frac{3}{8}\int_0^R \frac{GM^2(r)\,dr}{r^2} > \frac{GM^2}{8R} \tag{II.53}$$

By partially integrating the second integral in the middle of Eq. (II.53), we find

$$\int_0^R \frac{GM^2(r)\,dr}{r^2} = -\frac{GM^2}{R} + 2\int_0^R \frac{G\,M(r)\,dM(r)}{r} = -\frac{GM^2}{R} - 2E_G \qquad \text{(II.54)}$$

Substitute Eq. (II.54) into Eq. (II.53), add $\frac{3}{8}\frac{GM^2}{R}$, and use the virial theorem to eliminate $\int P\,dV$, we have

$$\pi P_c R^3 + \frac{3}{8}\frac{GM^2}{R} > -E_G > \frac{GM^2}{2R} \qquad \text{(II.55)}$$

Corollary. Eq. (II.48) can also be rewritten as:

$$\left(\frac{\rho_c}{\bar{\rho}}\right)^{4/3} \cdot \frac{3GM^2}{8\pi R^4} \geq P_c \qquad \text{(II.55a)}$$

Substitute this result into Eq. (II.55) we obtain

$$\frac{3}{8}\frac{GM^2}{R}\left[1 + \left(\frac{\rho_c}{\rho}\right)^{4/3}\right] > -E_G > \frac{GM^2}{2R} \qquad \text{(II.55b)}$$

Theorem 3. (Ritter). If radiation pressure is negligible and the gas is a classical ideal gas, that is,

$$P = \frac{k}{\mu m_p}\rho T \qquad \text{(II.56)}$$

where μ is the number of gas particles per proton mass (μ is the molecular weight $= \left(\frac{1}{A} + \frac{Z}{A}\right)^{-1}$) and if μ is a constant throughout the star, then

$$\bar{T} > \frac{1}{6}\frac{\mu m_p}{k}\frac{GM}{R} \qquad \text{(II.57)}$$

where $\overline{T}$ is the average temperature defined by

$$\overline{T} = \int_0^R T\,dM(r)\Big/M \tag{II.58}$$

<u>Proof.</u> Since

$$M\overline{T} = \int_0^R T\,dM(r) = \frac{\mu m_p}{k}\int_0^R \frac{P}{3}\,dM(r)$$

$$= \frac{\mu m_p}{k}\int_0^R P\,dV(r) = -\frac{1}{3}\frac{\mu m_p}{k}E_G \tag{II.59}$$

Use the right hand side of Eq. (II.55), we obtain

$$\overline{T} > \frac{1}{6}\frac{\mu m_p}{k}\frac{GM}{R} = 3.84\times10^6\,\mu\,\frac{M}{\odot}\frac{R_\odot}{R} \tag{II.60}$$

(A more stringent limit on E_G has been found by Chandrasekhar to be

$$\frac{3}{5}\frac{GM^2}{r_c} - E_G \geq \frac{3}{5}\frac{GM^2}{R} \tag{II.61}$$

where r_c is given by

$$M = \frac{4}{3}\pi \rho_c r_c^3 \tag{II.62}$$

Theorem 4. (Chandrasekhar). Let $(1-\beta)$ be the ratio of the radiation pressure to the total pressure, than if the gas is a classical ideal gas, then

$$(1-\beta_c) \leq 1-\beta^* \tag{II.64}$$

where β^* satisfies the quartic equation:

$$M = \left(\frac{6}{\pi}\right)^{1/2}\left[\left(\frac{k}{\mu_c m_p}\right)^4 \frac{3}{a}\,\frac{1-\beta^*}{\beta^{*4}}\right]^{1/2} \frac{1}{G^{3/2}} \tag{II.65}$$

Proof. The total pressure P is given by

$$P = \frac{k}{\mu m_p}\rho T + \frac{1}{3} a T^4 \tag{II.66}$$

where $\frac{1}{3} a T^4$ is the radiation pressure. The definition of $(1-\beta)$ is then

$$(1-\beta)P = \frac{1}{3} a T^4 ; \qquad \beta P = \frac{k}{\mu m_p}\rho T \tag{II.67}$$

From Eqs. (II.67) we obtain

$$T = \left[\frac{k}{\mu m_p}\,\frac{3}{a}\,\frac{1-\beta}{\beta}\right]^{1/3} \rho^{1/3} \tag{II.68}$$

$$P = \frac{1}{\beta}\frac{k}{\mu m_p}\rho T = \left[\left(\frac{k}{\mu m_p}\right)^4 \frac{3}{a}\,\frac{1-\beta}{\beta^4}\right]^{1/3} \rho^{4/3} \tag{II.69}$$

Hence

$$P_c = \left[\left(\frac{k}{\mu_c m_p}\right)^4 \frac{3}{a}\,\frac{1-\beta_c}{\beta_c^4}\right]^{1/3} \rho_c^{4/3} \tag{II.70}$$

From Eq. (II.48), we have

$$P_c \leq \frac{1}{2} G \left(\frac{4}{3} \pi \right)^{1/3} M^{2/3} \rho_c^{\,4/3} \tag{II.71}$$

With a little algebra, from Eqs, (II.70) and (II.71), we obtain

$$M \geq \left(\frac{6}{\pi} \right)^{1/2} \left[\left(\frac{k}{\mu_c m_p} \right)^4 \frac{3}{a} \, \frac{1-\beta_c}{\beta_c^{\,4}} \right]^{1/2} \frac{1}{G^{3/2}} \tag{II.72}$$

If we define β^* by Eq. (II.65), then clearly

$$\frac{1-\beta^*}{\beta^{*4}} \geq \frac{1-\beta_c}{\beta_c^{\,4}} \tag{II.73}$$

Since $(1-\beta)/\beta^4$ is a monotomic function of β we find

$$1-\beta^* \geq 1-\beta_c \tag{II,74}$$

The values of $(1-\beta^*)$ are 0.025, 0.1, 0.4, 0.6, 0.8 for $\left(\frac{M}{\odot}\right)\mu_c^{\,2}$ = 0.908, 2.130, 9.585, 26.41, 122.0.

II.6. The Uncertainty Principle of Heisenberg and the Energy-Density Relation of a Gas at low Temperature.

In some cases discussed previously it was assumed that the particle gas is a classical ideal gas. This assumption is valid only if the temperature is high, or the density is low. At high density and low temperature all real gases exhibit quantum phenomena. Among all quantum effects the most important one in astrophysics is the Fermi degeneracy.

The effect of Fermi degeneracy can be understood in terms of the Uncertainty Principle of Heisenberg. According to this principle, both the position $\underset{\sim}{r} = (x, y, z)$ and the momentum $\underset{\sim}{p} = (p_x, p_y, p_z)$ of a particle can be determined only to within an uncertainty $\Delta \underset{\sim}{r}$ and $\Delta \underset{\sim}{p}$ given by the following equations:

$$\Delta x \Delta p_x \geq \hbar$$
$$\Delta y \Delta p_y \geq \hbar$$
$$\Delta z \Delta p_z \geq \hbar \qquad \text{(II.75)}$$

The values of dynamical variables of a particle are eigen values of the wave equation in quantum mechanics, and the complete set of eigen values of the dynamical variables specify the <u>state</u> of the particle. The position, momentum, angular momentum, and so on, are typical dynamical variables. For a free electron there are three sets of dynamical variables: the position $\underset{\sim}{r}$, the momentum $\underset{\sim}{p}$ and the spin I $(=\frac{1}{2}\hbar)$. The spin has two eigen values corresponding to the two directions of the spin. Because of Eq. (II.75), if the spin of the electron is specified, states with different momentum and position coordinates within a volume element $\Delta^3 \underset{\sim}{r} \Delta^3 \underset{\sim}{p}$ (where $\Delta \underset{\sim}{r}$ and $\Delta \underset{\sim}{p}$ satisfies Eq. (II.75)) are not distinguishable and therefore should be counted as one state only. Since there are two value of spin allowed, within the volume element $\Delta^3 \underset{\sim}{r} \Delta^3 \underset{\sim}{p}$ there are two states.

Electrons are Fermions, hence will follow Pauli's exclusion Principle, which states that the number of particles in each state is either one or zero. This means that within the volume element $\Delta^3 \underset{\sim}{r}\, \Delta^3 \underset{\sim}{p}$ the maximum number of electrons is two.

Let r_e be the mean distance of separation between electrons, then the number density of electrons is

$$n_e = \frac{1}{\frac{4\pi}{3} r_e^3} \tag{II.76}$$

Let the mean uncertainty in the position of the electron be Δr_e. Certainly $\Delta r_e < r_e$. Also the mean uncertainty of the momentum, Δp_e is less than the average momentum p_e. Therefore

$$r_e p_e \geq \hbar \tag{II.77}$$

From Eq. (II.76) and (II.77) we obtain a maximum electron density $n_e^{(u.l.)}$ corresponding to an aberage electron momentum p_e:

$$n_e \qquad n_e^{(u.l.)} \approx \frac{g_e}{\hbar^3} \frac{3}{4\pi} \cdot p_e^3 \tag{II. 78}$$

where g_e is the statistical weight of the electron and $g_e = 2$. (A detailed analysis in Chapter III gives

$$n_e^{(u.l.)} = \frac{1}{\pi^2 \hbar^3} p_e^3 \tag{II.78a}$$)

For a given density Eq. (II.78) states that the average momentum p_e cannot fall below p_e. That is, even at absolute zero, microscopic kinetic energy of the electron is not zero but is some finite, the value of this residual thermodynamic energy E(T=0), is

$$E(T=0) = n_e \epsilon_e \tag{II.79}$$

where ϵ_e is approximately the mean energy:

$$\epsilon_e = (p_e^2 c^2 + m^2 c^4)^{1/2} - mc^2 \tag{II.80}$$

For non relativistic electrons

$$\epsilon_e \approx \frac{p_e^2}{2m} \tag{II.81}$$

Hence the energy density relation is (using the more exact relation (II.78a))

$$E(T=0) = n_e^{(u.l.)} \epsilon_e = \frac{(\pi^2 \hbar^3)^{2/3}}{2m} n_e^{5/3} \tag{II.82}$$

For relativistic electrons

$$\epsilon_e \approx p_e c \tag{II.83}$$

Hence the energy density relation is

$$E(T=0) = (\pi^2 \hbar^3)^{1/3} c \, n_e^{4/3} \tag{II.84}$$

For a real gas whose temperature is not zero, the energy-density relation is much more complicated. However if $kT << \epsilon_e$ Eqs. (II.82) and (II.84) are good approximations to the energy density relation. The limit $\frac{kT}{\epsilon_e} \to 0$ is known as the

Fermi degeneracy limit.

Because E(T=0) is inversely proportional to the mass, at $kT << \epsilon_e$ it is expected that the energy of the electrons will dominate over that of heavier particles. Hence at low temperature the pressure and energy of an electron gas dominates.

II.7. Dimension Analysis of Stellar Structure Equations. I. Fermi Degeneracy and Stellar Structure.

The equations of stellar structure are non linear equations; in general solutions cannot be generated from a known solution by a change of scale. Only in a few highly idealized cases is this possible.

However, for a star of homogeneous structure solutions show strong dependence on the mass, radius, temperature, and the equation of thermodynamic state. Approximate solutions can be obtained from a known solution by a change of scale; such an analysis of solution characteristics is known as "dimension analysis".

(a) The Gravitational Energy of a Star.

The upper limit for the gravitational energy of a star obtained in Eq. (II.55b) is a very generous upper limit. For most homogeneous stars the ratio of pressure to density follows a simple relation:

$$\frac{P}{\rho^{1+\frac{1}{n}}} = \text{Constant} \qquad \text{(II.85)}$$

It is found that the parameter n is roughly a constant (varying from 3 at the center to 1.5 at the surface for the sun). (n is known as the polytropic index). In chapter IX it is shown that if $P \propto \rho^{1+\frac{1}{n}}$ then the gravitational energy of a star is given by

$$E_G = -\frac{3}{5-n}\,\frac{GM^2}{R} \tag{II.86}$$

For other reasons the case $n \geq 5$ may be excluded from our discussion, and for most stars n varies from 1.5 to 3. Hence E_G is of the order of $\frac{GM^2}{R}$.

(b) Temperature, mass, radius relation for a star made of a classical ideal gas.

The pressure P is

$$P = \frac{1}{\mu}\,\frac{k}{m_p}\,\rho T$$

With the result from (a) the virial theorem then gives

$$\frac{3}{\mu}\,\frac{k}{m_p}\,\overline{T} = \frac{GM}{R} \tag{II.85}$$

Numerically

$$\overline{T} = \frac{M}{\odot}\,\frac{R_\odot}{R} = \left(\frac{M}{\odot}\right)^{2/3} \rho^{-1/3} \tag{II.86}$$

(c) Non-relativistic case of a Star Composed of Fermi Gas (The Maximum Temperature of Stars).

Express $\overline{T}$ in terms $\overline{P_e}$, we have, in the non-relativistic case,

$$k\overline{T} \approx \frac{\overline{P_e}^2}{2m} \tag{II.87}$$

We can express Eq. (II.86) in terms of $\overline{r_e}$ (the mean value of r_e throughout the star). We find,

$$\overline{\rho} = \frac{A}{Z} \frac{m_p}{\frac{4\pi}{3}\overline{r_e}^3}$$

$$\overline{r_e}\,\overline{T} = \frac{\mu}{3}\left(\frac{A}{Z}\right)^{1/3} \frac{G\, m_p^2\, N_p^{2/3}}{k} \qquad \text{(II.88)}$$

where N_p is the total number of proton masses of the star, A and z are the average values of the mass number and atomic number respectively.

Multiply both sides by $\overline{r_e}^2$ and use the uncertainty relation (II.77), we find:

$$2m\overline{r_e}^2 k\overline{T} \approx \overline{r_e}^2\, \overline{p_e}^2 \leq \hbar^2 \qquad \text{(II.89)}$$

Using Eq. (II.87) to eliminate $\overline{r_e}$, we find

$$\overline{T} \leq \left[\frac{\mu}{3}\left(\frac{A}{z}\right)^{1/3} \frac{G m_p^2 N_p^{2/3}}{\hbar\, k^{1/2}} (2m)^{1/2}\right]^2 = T_{max} \qquad \text{(II.90)}$$

Therefore T_{max} is the maximum temperature that a star of mass M may ever reach, ssuming that the electrons never become relativistic.

Express temperature in units of $T_0 = \frac{mc^2}{k} = 5.93 \times 10^9\ {}^0$K, Eq. (II.90) can be simplified to give

$$\overline{T} \leq T_{max} = 2T_0 \cdot \left[\frac{\mu}{3}\left(\frac{A}{z}\right)^{1/3}\right]^2 N_p^{4/3} \cdot \left(\frac{G m_p^2}{\hbar c}\right)^2 \qquad \text{(II.91)}$$

It is natural to define a unit N_0 for the number of proton masses in a star, that

$$N_0 = \left(\frac{Gm_p^2}{\hbar c}\right)^{-3/2} = 2.2034 \times 10^{57} \left(\simeq 2 N_\odot\right) \tag{II.92}$$

Eq. (II.92) then becomes

$$\bar{T} \leq T_{max} = 2 T_0 \left[\frac{\mu}{3}\left(\frac{A}{Z}\right)^{1/3}\right]^2 \left(N_p/N_0\right)^{4/3} \tag{II.93}$$

$\dfrac{Gm_p^2}{\hbar c}$ is the equivalent of the fine structure constant for the gravitational field. Numerically

$$\alpha_G = \frac{Gm_p^2}{\hbar c} = 5.9055 \times 10^{-39} \tag{II.94}$$

Eq. (II.93) can be used to understand the existence of planets and stars. The most abundant element in stars is hydrogen, whose ionization temperature T_i is roughly given by

$$kT_i \approx \frac{1}{2}\alpha^2 m_e c^2 = 13.6 \text{ ev}$$ (the ionization energy of the hydrogen atom)

$$T_i \approx 1.6 \times 10^5 \ {}^\circ K \tag{II.95}$$

Equating the numerical value of T_i to T_{max} and solve for (N_p/N_0), we find that if $M \lesssim 10^{-3} \odot$, the temperature of the star is never high enough to completely ionize hydrogen. <u>This may be taken to be the dividing mass between stars and planets.</u> The most massive planet in the solar system is the planet Jupiter, whose mass is around $10^{-3}\odot$. The smallest observed companion to a binary star is that of Barnard's star. The mass is calculated to be around $10^{-3}\odot$.

In order that hydrogen nuclear reaction may proceed favorably, the temperature of a star must be greater than $10^7\ {}^0K$. Use this value for T_{max}, we find it is necessary that

$$M \gtrsim 10^{-2}\odot \tag{II.96}$$

For stars of masses less than this value there is no nuclear energy source.

For stars of masses greater or equal to the solar mass, T_{max} is of the order of $6 \times 10^9\ {}^0K$. This is also the temperature for completion of nuclear reaction sequences until Fe^{56} is formed. Hence for stars of masses greater than the solar mass, elements upto Fe^{56} could be synthesized.

(d) Relativistic case of a star Composed of Fermi Gas. (The Landau-Chandrasekhar Mass Limit).

We now examine the case of large stellar mass, but will limit ourselves to the degeneracy limit. The energy density-pressure relation is

$$P = \frac{\alpha}{3} E \tag{II.97}$$

where $\alpha = 2, 1$ in the non-relativistic, and the relativistic cases respectively. The virial theorem then becomes

$$\alpha \int E\, dV = -E_G \approx \frac{GM^2}{R} \tag{II.98}$$

Neglecting the factor α which will introduce an error by at most a factor of 2, we can write approximately,

$$\int \varepsilon \, dV = N_p \bar{\epsilon}_e \frac{Z}{A} \qquad \text{(II.99)}$$

$$\bar{\epsilon}_e = \left[\bar{p}_e^{\,2} e^2 + m^2 c^4 \right]^{1/2} - mc^2$$

From the definition of $\bar{r}_e$, we find

$$R = \left(\frac{M}{m_p} \frac{Z}{A} \right)^{1/3} \bar{r}_e \qquad \text{(II.100)}$$

Multiply both sides of Eq. (II.98) by $\frac{1}{\hbar c}$, substituting Eqs. (II.99) and (II.100) into Eq. (II.98) and express N_p in units of N_0 (Eq. (II.92)), we find, in the limit $\overline{kT} << \bar{\epsilon}_e$

$$\left[\frac{N_p}{N_0} \left(\frac{A}{Z} \right)^2 \right]^{2/3} = \bar{r}_e \left\{ \frac{\left[\bar{p}_e^{\,2} c^2 + m^2 c^4 \right]^{1/2} - mc^2}{\hbar c} \right\}$$

$$\approx \left[1 + \left(\frac{\bar{r}_e}{\lambda_e} \right)^2 \right]^{1/2} - \frac{\bar{r}_e}{\lambda_e} \qquad \text{(II.101)}$$

where $\lambda_e = \hbar / mc$ = 3.8611x10 cm. is the Compton wave length of the electron. The Uncertainty relation (II.77) with the equal sign has been used to eliminate p_e.

The right hand side of Eq. (II.101) has a maximum at $\bar{r}_e = 0$, the maximum value is unity. For a given value of $\left(\frac{A}{Z} \right)^2 N_p < N_0$ it is possible to find a <u>positive</u> value of $\bar{r}_e$ such that Eq. (II.101) is satisfied. If $\left(\frac{A}{Z} \right)^2 N_p > N_0$, then no solution exists. Eq. (II.101) expresses the equilibrium condition of a star composed of a Fermi gas. The absence of a solution implies that equilibrium configurations does not exist if $\left(\frac{A}{Z}\right)^2 N_p \gtrsim N_0$ that is, if $\left(\frac{A}{Z}\right)^2 M \gtrsim 2\odot$. Since the pressure due to an electron gas dominates at low temperature, this means for stars of masses $M \gtrsim 2\alpha^{3/2}\left(\frac{Z}{A}\right)^2\odot$ no static equilibrium configuration exist. The existence of this mass limit was first predicted by Landau, but the numerical value for the limiting mass (M_{lim}) has been worked out by Chadrasekhar to be

$$\left(\frac{A}{Z}\right)^2 M_{lim} = 5.76\,\odot \tag{II.102}$$

At $\rho \gtrsim 10^4\, g/cm^3$ even at zero temperature hydrogen is not stable against nuclear reaction leading to the formation of heavier elements, hence the ratio of Z/A is around 1/2. This means that the limiting mass is around 5.76/4 $\odot$ = 1.44$\odot$. In practice β decay instability will set in for masses between $\odot$ and 1.4$\odot$, depending on the composition, Hence the practical mass limit is less than 1.44$\odot$.

Our analysis also shows that r_e is of the order of λ_e. This will give a mean density of around $10^5 \rightarrow 10^6$ g/cm^3 . This value is consistent with observed values for the density of white dwarfs.

At finite temperature the $\sim$ sign in Eq. (II.101) is replaced by $>$, and our conclusion is not valid, equilibrium configuration for stars of arbitrary mass exist. However, the energy supply of all stars is limited. Eventually the star has to encounter the mass limit problem. This problem is discussed in Chapter XIII.

II.8. Dimension Analysis of Stellar Structure Equations.

II. Radiation and Stellar Structure.

(a) Stability of Massive Stars.

According to Eq. (II.12) the ratio α of the total energy to the gravitational energy of a star decreases as P approaches (1/3)E. If a small amount of energy δW is supplied to the star, the gravitational energy of the star will change by a fraction $\frac{\delta E_G}{E_G} \sim \frac{\delta W}{W}$. This means that the radius of the star will also change approximately by $\frac{\delta R}{R} \sim \frac{\delta W}{W}$. As α approaches zero, w approaches zero and the supply of an energy much smaller than E_G but comparable to w, can still cause the radius of the star to change considerably. The star is thus easily made to oscillate.

The oscillation can take place spontaneously if the star has an energy source that is strongly temperature dependent. When a small oscillation takes place, when the radius is at its minimum the temperature of the star will be at its maximum causing energy to be over produced. This energy will further increase the amplitude of oscillation, this in turn increases the energy supply. When oscillations of large amplitudes take place, matter will be ejected from the star, reducing its mass.

For radiation $P = (1/3)E$, hence when the radiation pressure dominates over the gas pressure the star will become unstable against oscillation. We may therefore take as the condition for instability, when the ratio

$$\frac{\text{radiation pressure}}{\text{gas pressure}} \approx \frac{\text{radiation energy density}}{\text{gas energy density}}$$

becomes too large then the condition for stability against pulsation is that

$$\mathcal{R} = \left(\frac{\text{radiation energy density}}{\text{gas energy density}}\right)_{\text{average}} \qquad \text{(II.103)}$$

must not exceed some value of the order one.

Approximate the average radiation energy density by $a\overline{T}^4$. This approximation will over emphasize the importance of the radiation energy density, resulting in too large an estimate of $\mathcal{R}$. Hence the condition for stability against pulsation becomes that $\mathcal{R}$ must not be too large.

The average gas energy density is (for a non- relativistic and non-degenerate gas)

$$\frac{\left(1+\frac{1}{Z}\right)\frac{3}{2}k\overline{T}}{\frac{4\pi}{3}\overline{r_e}^3} \qquad \text{(II.104)}$$

The term (1/Z) is due to contributions from nuclei. Use Eq. (II.87) to eliminate $\overline{r_e}$, and from the definition of a (Steffan-Boltzmann constant)

$$a = \frac{\pi^2 k^4}{15c^3\hbar^3} \qquad \text{(II.105)}$$

we find

$$\mathcal{R} = \frac{8\pi^3}{135}\,\frac{1}{27}\,\mu^4\left(\frac{N_p}{N_0}\right)^2 = \frac{\mu^4}{30}\left(\frac{M}{M_\odot}\right)^2 \qquad \text{(II.106)}$$

Therefore $\mathcal{R}$ increases as the square of M. If we restrict ourselves to not too large a value of $\mathcal{R}$, then there is an upper mass limit for the stars. A detailed analysis by Harm and Schwarzschild indicates an upper limit of $60\odot$. This is consistent with the maximum mass observed. (The Plaskett's Star).

(b) Stellar Luminosity.

Energy produced inside a star diffuses outwards to the surface where it is radiated away. Nuclear energy is the main energy source of a star, and the nuclear energy production rate is very temperature sensitive. If the following approximation is used for the production rate $\mathcal{E}$ (ergs per g-sec) at constant density,

$$\mathcal{E}_n = \mathcal{E}_n^{(0)} T^k \tag{II.107}$$

where $\mathcal{E}_n^{(0)}$ is a constant, then the exponent k is around 5 for the proton-proton reaction and is about 10 for other reactions. The rate of outflow of energy is generally not a sensitive function of the temperature, being roughly proportional to T (assuming a constant opacity.) The opacity does not vary as drastically with the temperature as reaction rate does. Hence the nuclear the production rate can be caused to vary by orders of magnitude by a small change in temperature while the outflow rate remains practically constant. This means that the temperature of a star is adjusted until the rate of production rate equals the outflow rate. If a slight temperature perturbation is applied to a star, the star will tend to restore its temperature to the value at which a balance of energy is reached. This can be seen as follows: Consider the case when the production rate exceeds the outflow rate. Energy will accumulate in the star, increasing the temperature. Then the pressure inside will increase.

An increase in the pressure will expand the star, decreasing its temperature, resulting in a decrease in the energy production rate; the temperature as well as the pressure at the center will decrease and the star will contract, heating up the star and increasing the energy production rate.

While a detailed knowledge of stellar luminosity requires a full treatment of the radiative transfer problem, a rough estimate can be obtained from physical arguments. The rate of radiative transport is dependent on the interaction of photons with matter, and the problem may be regarded as a diffusion problem. The rate of diffusion is determined by the frequency absorption and re-emission takes place. If there were no absorption, photons emitted at the center of a star will reach the surface in R/c seconds. After a photon is absorbed and re-emitted,the direction of the re-emitted photon is not the same as the original photon. Hence the diffusion problem can be regarded as a random walk problem: a photon changes its direction after one mean free path. According to the theory of random walk problem, if the mean free path is Λ , then the number of collisions N_Λ a particle undergoes before it travels a distance R is given by

$$N_\Lambda = \left(\frac{R}{\Lambda}\right)^2 \tag{II.108}$$

The time it taks for the particle to travel the distance R is given by

$$t_\Lambda = \frac{N_\Lambda \Lambda}{v} = \frac{R^2}{\Lambda}\frac{1}{v} \tag{II.109}$$

where v is the velocity. Applying this result to a star, the luminosity L is given by

$$L = \frac{\text{Total radiation energy of the star}}{t} \tag{II.110}$$

Approximate the total radiation energy by $\frac{4\pi}{3} R^3 a \overline{T}^4$.
The mean free path Λ is $1/\sigma n$ where σ is the absorption or scattering cross-section and n is the number density of absorbers or scatterers. <u>Assume</u> that the number density of scatterers and absorbers is the same as the electron number density, we find

$$n = \frac{1}{\frac{4\pi}{3}\bar{r}_e^{\,3}} \tag{II.111}$$

Eliminating $\bar{T}$ by using Eq. (II.87), we find

$$L = \frac{16\pi^4}{135}\left(\frac{\lambda_e^2}{\sigma}\right)\frac{m^2c^4}{\hbar}N_0^{1/2}\left(\frac{N}{N_0}\right)^3\left(\frac{\mu}{3}\right)^4\left(\frac{A}{Z}\right)^{4/3} \tag{II.112}$$

Define the unit time t_0 and the unit of stellar luminosity L_0 and the unit of cross section σ_0 by the following equations:

$$t_0 = \frac{\frac{\hbar}{mc}\,\frac{1}{c}}{\alpha_G} = 2.1811\times 10^{17}\,\text{sec} = 6.91\times 10^{9}\,\text{years} \tag{II.113}$$

$$L_0 = \frac{N_0 mc^2}{t_0} = 8.309\times 10^{33}\,\frac{\text{ergs}}{\text{sec}} \tag{II.114}$$

$$\sigma_0 = \lambda_e^2 = (\hbar/mc)^2 = 4\times 10^{-21}\,\text{cm}^2 \tag{II.115}$$

Expressing L in terms of L_0 and t_0, we find

$$\begin{aligned} L &= \frac{16\pi^4}{135}\left(\frac{\mu}{3}\right)^4\left(\frac{A}{Z}\right)^{4/3}\cdot L_0\left(\frac{N}{N_0}\right)^3\left(\frac{\lambda_c^2}{\sigma}\right) \\ &\cong 4\left(\frac{\mu}{3}\right)^4\left(\frac{A}{Z}\right)^{4/3}\left(\frac{\sigma_0}{\sigma}\right)^2 L_\odot\left(M/\odot\right)^3 \end{aligned} \tag{II.116}$$

According to our simple theory, $L \propto \left(\frac{\sigma_0}{\sigma}\right)^2\left(\frac{M}{\odot}\right)^3$.

The cross section σ depends on the temperature as well as on the density. Since at a given temperature the density depends on the mass of a star, the cross section σ depends on the mass indirectly and the resulting mass luminosity law is $L \propto M^n$ where n is close to 3.

II.9. Evolutionary Time Scale of the Stars.

The main energy source of a star is the nuclear energy. The time t_n for a star to consume its nuclear energy is

$$t_n = \int_M^0 \frac{dE(M_n)}{L(M_n)} \tag{II.117}$$

where $E(M_n)$ is the nuclear energy content corresponding to the unburnt nuclear fuel of mass M_n, and $L(M_n)$ is the luminosity at M_n. For hydrogen reactions $E(M_n) = 0.0007c^2M_n$, we have

$$t_n = 0.0007c^2 \int_M^0 \frac{dM_n}{L(M_n)} \tag{II.118}$$

The luminosity of a star is reasonably constant until a fraction x_n of its total available nuclear fuel is used up, afterwards the structure of the star changes drastically and the luminosity increases rapidly. (Details of this are discussed in Chapter XI.) Hence we can approximate Eq. (II.118) by the following expression:

$$t_n = 0.001c^2 \frac{x_n M}{L}$$

The value of x_n is 0.1 for stars like the sun but may be as high as 0.5 for stars whose masses are in excess of 10 $\odot$. Using Eq. (II.116) for L, we find

$$t_n \approx 9\times10^{10} x_f \left(\frac{\sigma}{\sigma_0}\right)\left(\frac{\odot}{M}\right)^2$$

For the sum $x_f = 0.1$, $\sigma \sim \sigma_0$ and $t_n \sim 9\times10^9$ years. Detailed calculation gives a value of 10^{10} years. For stars of masses $\sim 30\ \odot$

the life time is only around 10^6 years. Hence all massive stars we now observe are relatively young Population I stars, newly condensed from the interstellar medium. According to the current theory of the origin of the elements, all chemical elements are synthesized in stellar nuclear reactions and redistributed into space by supernova explosion or steady mass loss processes. This explains why massive stars are rich in heavy elements.